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AN INVARIANCE PRINCIPLE FOR MIXINGALE

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Abstract

In this paper, an invariance principle for mixingales is established using a method of proof different from that employed in McLeish (1975, 1977), Wooldridge (1986) and Gallant (1987).

1. Introduction

In his 1975 and 1977 papers, McLeish established invariance principles (IPs) for mixingales. There has not been much development in this area since then. It was only recently that Wooldridge (1986) considered mixingales and developed an IP for functions of dependent variables a topic originally considered by McLeish (1975).

A classical procedure used in establishing an IP is to verify the conditions of a Wiener process. Wooldridge (1986) and Gallant (1987), Drogyn (1971), McLeish (1974) and Herrndorf (1984), to mention a few, provide examples of this latter procedure. Another approach, given in Ethier and Kurtz (1986) (see Aldous, 1989), starts with a characterization of the limit process and shows that the characterization is asymptotically true for the approximating processes.

In this paper an alternative tool is used to obtain an asymptotic distribution for mixingales. McLeish (1975) employed two conditions in the unconditional and conditional expectation of the squared sums in order to obtain an IP. In this paper, no assumption on the conditional expectation was used. This was made possible through the use of Andrews' (1988) result on correlation.

The organization of this paper is as follows. In Section 2, a result on IP is established using the approach of McLeish (1974). A CLT is then obtained as a special case. The proofs are given in Section 3.

2. Main Result

A sequence (X_i, F_i) is an L^2-mixingale if there exist non-negative constant $(c_i : i \geq 1)$ and $(\Psi_m : m \geq 0)$ such that for all $i \geq 1$ and $m \geq 0$ we have

(a) $\| E_{i-m} X_i) \|_2 \leq c_i \Psi_m$ and

(b) $\| X_i - E_{i+m} X_i \|_2 \leq c_i \Psi_{m+1}$

We will consider the following random variable

$$(1) \quad W_n(a) = n^{-1/2} \sum_{i=1}^{[na]} X_i , \quad a \in [0,1] ,$$

where $\{X_1, F_i\}$ is a L^2-uniformly integrable L^2-mixingale with mixingale numbers $\{\Psi_m\}$ and constants $\{c_i\}$ such that $\Psi_m < Bm^{\theta}$, $\theta < -1$, and $\lim_{n\to\infty} \sum_{i=1}^{n} c_i < \infty$. It is easily seen that $n^{-1/2} S_n$ is just a special case of (1) with $a = 1$, i.e. $n^{-1/2}$ Sn $= W(1)$. Here $W(a)$ is a random element in $D[0,1\}$, where $D[0,1]$ is the space of right continuous functions with left and right limits on the closed interval $[0,1]$.

In the proof we use the following Taylor expansion:

$$(2) \quad \log_e (1+x) = x - x^2/2 + r(x)$$

where $|r(x)| < x^3$, so that

$$(3) \quad (1+x) \exp[x^2/2 + r(x)] = \exp[x].$$

Therefore, substituting $x = itx_i$ in (3), we obtain

$$(4) \quad \prod_{i=1}^{n} (1+itx_i) = \prod_{i=1}^{n} \exp[itx_i - (itx_i)^2/2 + r(itx_i)\}$$

$$= \exp[\sum_{i=1}^{n} \{(itx_i) + t^2x_i{}^2/2 + r(itx_i)\}]$$

or

$$(5) \quad \prod_{i=1}^{n} (1+itx_i) \exp[\sum_{i=1}^{n} \{(-t^2x_i{}^2/2 + r(itx_i)\}]$$

$$= \exp[\sum_{i=1}^{n} itx_i].$$

Taking the expectation of both sides

$$(6) \quad E \prod_{i=1}^{n} (i + itx_i) \sum_{i=1}^{n} \exp[(-t^2x_i{}^2/2 + r(itx_i)]$$

$$= E \exp[\sum_{i=1}^{n} itx_i],$$

where the RHS is the cf of $\sum_{i=1}^{n} x_i$.

THEOREM 1. Suppose $\{X_i, F_i\}$ is a uniformly integrable L^2-mixingale with $\Psi_m \le Bm^\Theta$, $\Theta < -1$, and $\lim_{n=>\infty} \sum_{i=1}^{n} c_i < \infty$. Suppose further that,

(a) $\max_j |n^{-1/2} X_j| ->_p 0$ as $n -> \infty$

(b) $n^{-1} \sum_{j=1}^{[na]} X_j^2 ->_p a$ as $n -> \infty$.

Then, $W_n(a)$ converges weakly to the Wiener process $W(a)$. ▫

As noted by McLeich (1974), condition (a) and (b) in Lemma 1 and Theorem 1 are weaker versions of Billingsley's (1968, Theorem 19.2) condition. Therefore, Theorem 1 imposes weaker conditions than those of McLeish (1975, 1977).

The condition that $\Psi_m \leq Bm^{\theta}$, $\theta < -1$, is not unusual. Wooldridge (1986), Gallant (1987) and Gallant and White (1988) also use the same condition with $\theta < -1/2$. This seems to be the trade-off in using the method of McLeish (1974). However, we believe that this approach can be used to deal with correlation directly, without need of a mixingale assumption.

COROLLARY 1.

Suppose $\{X_i, F_i\}$ is a uniformly integrable L^2-mixingale such that the conditions of Theorem 1 are satisfied. Then, $n^{-1/2}S_n$ is asymptotically normally distributed with mean 0 and variance 1. •

3. Proofs

To prove Theorem 1, we need the following lemmas. First, we need the result of Gallant (1987) [see also McLeish, 1975, 1977] showing that

$$(7) \quad P_n(A) = P(\omega: W_n(.) \in A).$$

is tight.

LEMMA 1. (Gallant, 1987). Let $\{X_i, F_i\}$ be a uniformly integrable L^2-mixingale. Suppose that $\Psi_m \leq Bm^{\theta}$, $\theta < -1/2$, and $\lim_{n\to\infty} \sum_{i=1}^{n} c_i < \infty$. Then $\{P_n\}$ is tight. Furthermore, if $P_n(A) = P\{\omega: W_n(.) \in A\}$ and if P is a limit distribution of P_n, P puts mass one in C[0,1].

PROOF. See Gallant (1987).

By the results of Lemma 1, $P_n(A)$ is tight and if P is a limit distribution of P_n, P puts mass one in D[0,1]. by Proposition 1.2 of Aldous (1989). $P_n \Rightarrow_w P$. Therefore, in this section we only need to establish convergence of finite dimensional distributions of P_n, since if the finite dimensional distributions P_n converge to P and $\{P_n\}$ is tight, then $P_n \Rightarrow_w P$ [see also Billingsley, 1968].

The following lemma, Lemma 2, is more general version of Andrews' (1988) result on the correlation of X_i and X_j. The lemma plays an important role in proving the finite dimensional limit distribution of (1).

LEMMA 2. Let (X_i, F_i) be an L^2-uniformly integrable L^2-mixingale. For any integer $s \in [0, (i-j)]$, we have

$$(8) \quad |EX_iX_j| \leq (\Psi_0 + \Psi_1)\Psi_{s+1}c_ic_j + (\Psi_{s+1} + \Psi_0 + \Psi_1)\Psi_{(i-j-s)}c_ic_j$$

In particular, taking $s = (1-j)/2$

$$(9) \quad |EX_iX_j| \leq 2(\Psi_0 + \Psi_1 + \Psi_{[(i-j)/2]+1})\ \Psi_{[(i-j)/2]}c_ic_j.$$

PROOF:

We follow Andrews (1988). For any integer $s \in [0, i-j]$, we have

$$(10) \quad |EX_i\ X_j| \leq |EX_i\ (X_j - E_{j+s}X_j)| + |EX_iE_{j+s}X_j|$$
$$= |EX_i\ (X_j - E_{j+s}X_j)| + |EE_{j+s}(X_iE_{j+s}X_j)|$$

Using the Cauchy-Schwarz inequality,

$$|EX_iX_j| \leq \|X_i\|_2X_j - E_{j+s}X_j\|_2 + \|E_{j+s}X_i\|_2\|E_{j+s}X_j\|_2$$

Using the definition of mixingale,

$$(11) \quad |EX_iX_j| \leq (\Psi_0 + \Psi_1)\ \Psi_{s+1}c_ic_j + (\Psi_{s+1} + \Psi_0 + \Psi_1)\Psi_{(i-j-s)}c_ic_j$$

which yeilds (8). Taking $s = (i-j)/2$,

$$(12) \quad |EX_iX_j| \leq (\Psi_0 + \Psi_1)\Psi_{[i-j)/2]+1}c_ic_j + (\Psi_{[(i-j)/2]+1} + \Psi_0 + \Psi_1)\Psi_{[(i-j)/2]}c_ic_j.$$

Since $\Psi_{[(i-j)/2]+1} \leq \Psi_{[(i-j)/2]}$, (12) gives (9). This completes the proof. ▪

Now we show that $W_n(a)$ has an asymptotic normal distribution. We do this through the method of cf's using the same technique used McLeish (1974).

LEMMA 3. Suppose $\{X_i, F_i\}$ is an L^2-uniformly integrable L^2-mixingale, $p \geq 2$, such that $\lim_{n\to\infty} \sum_{i=1}^{n} c_i < \infty$ and $\Psi_m \leq B\ m^{\theta}$, for some $B < \infty$ and $\theta < -1$. Define

(13) $$T_n = \prod_{i=1}^{n} (1 + n^{-1/2} itX_i).$$

If

(a) $$\max_{j\geq 1} |n^{-1/2} X_j| \to_p 0$$

(b) $$n-1 \sum_{j=1}^{n} x_j^2 \to_p a \text{ as } n\to\infty$$

are satisfied, then T_n is uniformly integrable and

(14) $$\lim_{n\to\infty} E\, T_n = 1.$$

PROOF:

Performing the multiplication in (3), we have

(15) $$\prod_{i=1}^{n} (1 + n^{-1/2} itX_i) - 1$$

$$= n^{-1/2} \sum_{i=1}^{n} itX_i - n^{-1} \sum_{\substack{j>i \\ i=1}}^{n} t^2 X_i X_j + o(n^{1+\delta})$$

Therefore,

(16) $$\left|E \left[\prod_{i=1}^{n} (1 + n^{-1/2} itX_i)\right] - 1\right|$$

$$= \left|n^{-1/2} E \left[\sum_{i=1}^{n} itX_i \right] - n^{-1} E\left[\sum_{\substack{j>i \\ i=1}}^{n} t^2 X_i X_j\right] + o(n^{1+\delta})\right|$$

$$\leq n^{-1/2} it \sum_{i=1}^{n} |EX_i| + n^{-1} t^2 \sum_{\substack{j>i \\ i=1}}^{n} |EX_i X_j| + o(n^{1+\delta})$$

Since $EX_i = 0$, the first expression of the RHS is 0. For the second expression of the RHS, using Lemma 2,

$$(17) \quad \sum_{\substack{j>i \\ i=1}}^{n} |EX_i X_j|$$

$$\le 2 \sup_{k\ge1} \|X_k\|_2^2 \sum_{\substack{j=i \\ i=1}}^{[n/2]} (\Psi_0 + \Psi_1 + \Psi_{[(j-i)/2]}) \Psi_{[j-i)/2]}$$

$$\le 2 \sup_{k\ge1} \|X_k\|_2^2 \, n \, (\sum_{u=0}^{[n/2]} (\Psi_0 + \Psi_1) \Psi_u + \sum_{u=0}^{[n/2]} \Psi_u^2).$$

Approximating the sum by its integral, and using assumption $\Psi_u \le B\, u\theta$,

$$(18) \quad n^{-1} t^2 \sum_{\substack{j>1 \\ i=1}}^{n} E|X_i X_j|$$

$$\le 2\, t^2 \sup_{k\ge1} \|X_k\|_2^2 \, (\Psi_0 + \Psi_1)\, B' [n/2]^{\theta+1}$$

$$+ 2\, t^2 \sup_{k\ge1} \|X_k\|_2^2 \, B'' [n/2]^{2\theta+1} + o(n^{1+\delta})$$

Since $\theta < -1$, the last inequality converges in probability to 0 as $n \to \infty$. Therefore, (18) converges to 0 as $n \to \infty$ and, hence, (14) holds. The uniform integrability of T_n follows from the uniform integrability of $\{X_i^2\}$ and the fact that $E\, T_n$ is bounded by 1. ▫

LEMMA 4. (McLeish, 1974). Let

$$(19) \quad T_n = \prod_{i=1}^{[na]} (1 + itX_i)$$

Suppose for all real t,

(a) $ET_n \to 1$,

(b) $\{T_n\}$ is uniformly integrable,

(c) $n^{-1} \sum_{i=1}^{[na]} X_i^2 \to_p a$, and

(d) $\max_{j \le n} n^{-1/2}|X_j| \to_p 0$.

Then $S_n = \sum_{i=1}^{n} X_i \Rightarrow_w N(0,1)$

PROOF: See McLeish (1974). •

LEMMA 5. Suppose $\{X_i, F_i\}$ is a uniformly integrable L^2-mixingale with $\Psi_m < Bm^\theta$, $\theta < -1$, and $\lim_{n \Rightarrow \infty} \sum_{i=1}^{n} c_i < \infty$. Suppose further that,

(a) $\max_{j \ge 1} n^{-1/2} |X_j| \to_p 0$ as $n \to \infty$

(b) $n^{-1} \sum_{j=1}^{[na]} X_j^2 \to_p a$ as $n \to \infty$.

Then, $W_n(a)$ is asymptotically nomrally distributed with mean 0 and variance a.

PROOF: Note that Lemma 3 still holds if n is replaced by [na]. Therefore, defining

$$T_n(a) = \prod_{i=1}^{[na]} (1 + n^{-1/2} itX_i)$$

we have $E\, T_n(a) \to 1$ and $T_n(a)$ is uniformly integrable. Now we are led to verify the conditions of Lemma 4. However, the remaining conditions of Lemma 4 are just the assumptions (a) and (b) of Lemma 5. And, therefore, $W_n(a)$ converges to $N(0,a)$. •

LEMMA 6. Suppose $\{X_i, F_i\}$ is a uniformly integrable L^2-mixingale with $\Psi_m \le Bm^\theta$, $\theta < -1$, and $\lim_{n \Rightarrow \infty} \sum_{i=1}^{n} c_i < \infty$. Suppose further that,

(a) $\max j\, n^{-1/2} X_j \to_p 0$ as $n \to \infty$

(b) $n^{-1} \sum_{j=1}^{[na]} X_j^2 \to_p a$ as $n \to \infty$.

Then, $W_n(a)$ has asymptotic independent increments.

PROOF:

Define

(20) $$T_n(a) = \prod_{i=1}^{[na]} (1 + n^{-1/2} its_1 X_i)$$

(21) $$I_n(a) = \exp[its_1 W_n(a)]$$

(22) $$U_n(a) = \exp[-(1/2)t^2(n^{-1} \sum_{i=1}^{[na]} s_1^2 X_i^2 + \sum_{i=1}^{[na]} r(n^{-1/2} its_1 X_i)],$$

where r(.) is the reminder defined as in (2),

(23) $$T_n(b,c) = \prod_{i=[nb]}^{[nc]} (1 + n^{-1/2} its_2 X_i)$$

(24) $$I_n(b,c) = \exp[it(s_2(W_n(c) - W_n(b)))]$$

(25) $$U_n(b,c) = \exp[-(1/2)t^2(n^{-1} \sum_{i=[nb]+1}^{[nc]} s_2^2 X_i^2) + \sum_{i=[nb]+1}^{[nc]} r(n^{-1/2} its_2 X_i)].$$

Following McLeish (1974), in order that $(s_1 W_n(a) + s_2(W_n(c) - W_n(b))$ will have an asymptotically normal distribution with mean 0 and variance $s_1^2 a + s_2^2(c-b)$, we have to show:

(26) $$ET_n(a)T_n(b,c) \to 1$$

(27) $T_n(a)T_n(b,c)$ is uniformly integrable

(28) $$T_n(a)T_n(b,c)(U_n(a)U_n(b,c) - \exp[-(1/2)t^2(s_1 a + s_2(c-b))] \xrightarrow[L^1]{} 0.$$

To prove (26) and (27), note that

(29) $$|E\,(T_n(a)T_n(b,c) - 1)|$$

$$= \left|E\left[n^{-1/2}\sum_{i=1}^{[na]} its_1X_i - n^{-1}\sum_{\substack{i=1\\ j>1}}^{[na]} t^2s_1^2X_iX_j\right.\right.$$

$$+ n^{-1/2}\sum_{i=[nb]+1}^{[nc]} its_2X_i - n^{-1}\sum_{\substack{i=[nb]+1\\ j>1}}^{[nc]} t^2s_2^2X_iX_j$$

$$\left.\left. - n^{-1}\,t^2\sum_{i=1}^{[na]}\sum_{j=[nb]+1}^{[nc]} s_1s_2X_iX_j + o(n^{1+\delta})\right]\right|$$

$$\le n^{-1/2}\sum_{i=1}^{[na]} its_1|Ex_i| + n^{-1}\sum_{\substack{i=1\\ j>1}}^{[na]} t^2s_1^2\,|EX_iX_j|$$

$$+ n^{-1/2}\sum_{i=[nb]+1}^{[nc]} its_1|EX_i| + n^{-1}\sum_{\substack{i=[nb]+1\\ j>i}}^{[nc]} t^2s_2^2|Ex_iX_j|$$

$$+ n^{-1}\,t^2\sum_{i=1}^{[na]}\sum_{j=[nb]}^{[nc]} s_1s_2|Ex_iX_j| + o(n^{1+\delta})$$

Since $EX_i = 0$, $n^{-1/2}\sum_{i=1}^{[na]} its_1|EX_i|$ and $n^{-1/2}\sum_{i=[nb]}^{[nc]} its_2|EX_i|$ are 0. Now,

(a) To show:

$$n^{-1}\sum_{\substack{i=1\\ j>1}}^{[na]} t^2s_1^2|EX_iX_j| \;\to\; 0\ .$$

Note that from Lemma 2 with $s = (i-j)/2$,

$$(30)\quad \sum_{\substack{j>i\\ i=1}}^{[na]} |EX_iX_j|$$

$$\le 2 \sup_{k\ge1} \|X_k\|_2^2 \sum_{\substack{j>i\\ i=1}}^{[na]} (\Psi_0+\Psi_1+\Psi_{[(j-i)/2]})\Psi_{[(j-i)/2]}$$

$$\le 2 \sup_{k\ge1} \|X_k\|_2^2\, n \Big(\sum_{u=0}^{[[na]/2]} (\Psi_0+\Psi_1)\Psi_u + \sum_{u=0}^{[[na]/2]} \Psi_u^2 \Big).$$

By the assumption, we have $\Psi_u \le B\, u^{\theta}$. Therefore, using the integral approximation to the sum, we obtain

$$(31)\quad \sum_{u=0}^{[[na]/2]} \Psi_u \le B'(n/2)^{\theta+1}$$

and

$$(32)\quad \sum_{u=0}^{[[na]/2} \Psi u^2 \le B''\, (n/2)^{2\theta+1}$$

where $B' = B\,(1+\theta)^{-1}$ and $B'' = B\,(1+2\theta)^{-1}$. Substituting (31) and (32) in (30),

$$(33)\quad n^{-1}\, t^2 \sum_{\substack{j>1\\ i=1}}^{[na]} |EX_iX_j|$$

$$\le 2\, t^2 \sup_{k\ge1} \|X_k\|_2^2\, (\Psi_0+\Psi_1)\; B'\; (n/2)^{\theta+1}$$

$$+\; 2\, t^2 \sup_{k\ge1} \|X_k\|_2^2\;\; B''(n/2)^{2\theta+1}$$

Since $\theta < -1$, the last inequality converges to 0 as $n \to \infty$.

(b) To show:

$$(34)\quad n^{-1} \sum_{\substack{i=[nb]+1\\ j>1}}^{[nc]} t^2 s_2^2 |EX_iX_j| \to 0\,.$$

Note that, since $|EX_iX_j| \geq 0$,

$$(35) \quad \sum_{\substack{i=[nb]+1 \\ j>i}}^{[nc]} |EX_iX_j| \leq \sum_{\substack{i=1 \\ j>i}}^{[nc]} |EX_iX_j|$$

which goes to 0 with the same argument as in (30) - (33) of the proof in (a) by replacing [na] by [nc].

(c) To show:

$$n^{-1}t^2 \sum_{i=1}^{[na]} \sum_{j=[nb]+1}^{[nc]} s_1s_2|EX_iX_j| \to 0.$$

Note, that

$$(36) \quad \sum_{i=1}^{[na]} \sum_{j=[nb]+1}^{[nc]} |EX_iX_j| \leq \sum_{\substack{i=1 \\ j>1}}^{[nc]} |EX_iX_j|.$$

Therefore $n^{-1}t^2 \sum_{i=1}^{[na]} \sum_{j=[nb]+1}^{[nc]} s_1s_2|EX_iX_j|$ also converges to 0 by similar arguments as in (b).

Using the results of (a)-(c), $n^{-1} \sum_{\substack{i=1 \\ j>i}}^{[na]} t^2s_1^2|EX_iX_j|$, $n^{-1} \sum_{\substack{i=[nb] \\ j>1}}^{[nc]} t^2s_2^2|EX_iX_j|$, $n^{-1}t^2 \sum_{i=1}^{[na]} \sum_{j=[nb]}^{[nc]} s_1s_2|EX_iX_j|$ converge to 0. This implies that $E(T_n(a)T_n(b,c) \to$ as $n \to \infty$.

The uniform integrability of $\{T_n(a)T_n(b,c)$ follows from the uniform integrability of $\{X_i^2\}$ and the fact that $E(T_n(a)T_n(b,c)$ is bounded by 1.

To show (28), note that

(37) $$\sum_{i=1}^{[na]} r(n^{-1/2} its_1 X_i) \le \sum_{i=1}^{[na]} |n^{-1/2} its_1 X_i|^3$$

$$\le \sum_{i=1}^{[na]} |\max n^{-1/2} X_i| \; |its_1| \; |n^{-1/2} its_1 X_i|^2$$

which converges in probability to 0 by (a) and (b). Similarly,

(38) $$\sum_{i=[nb]+1}^{[nc]} r(n^{-1/2} its_2 X_i) \le \sum_{i=[nb]+1}^{[nc]} |n^{-1/2} its_2 X_i|^3$$

$$\le \sum_{i=[nb]+1}^{[nc]} |\max n^{-1/2} X_i| \; |its_2| \; |n^{-1/2} its_2 X_i|^2$$

converges in probability to 0 by (a) and (b). Equations (37), (38) and assumption (b) imply that

(39) $$U_n(a)U_n(b,c)$$

$$= \exp[-(1/2)t^2(n^{-1} \sum_{i=1}^{[na]} s_1^2 X_i^2)]$$

$$\times \exp[-(1/2)t^2(n^{-1} \sum_{[nb]+1}^{[nc]} s_2^2 X_j^2)]$$

$$\times \exp[\sum_{i=1}^{[na]} r(n^{-1/2} its_1 X_i)]$$

$$\times \exp[\sum_{i=[nb]+1}^{[nc]} r(n^{-1/2} its_1 X_i)]$$

converges in probability to 0. The uniform integrability of

$T_n(a)T_n(b,c)(U_n(a)U_n(b,c) - \exp[-(1/2)t^2(s_1^2 a + s_2^2(c-b))] = (I_n(a)I_n(b,c) - T_n(a)T_n(b,c)\exp[-\exp[-(1/2)t^2(s_1 a + s_2(c-b))]$ follows from the boundedness of $\exp[itx]$ and $ET_n(a)T_n(b,c)$. Therefore,

(40) $$T_n(a)T_n(b,c)U_n(a)U_n(b,c)$$

$$\longrightarrow_{L^1} \exp[-(1/2)t^2(s_1^2 a + s_2^2(c-b))].$$

Having proven (26), (27), (28), we have completed the proof of Lemma 6.

PROOF OF THEOREM 1. By Lemma 5 and 6, $W_n(0)$ has limiting normal distribution $W(a)$ having mean 0 and variance a. Furthermore, for $0 = a_1 < b_1 \le a_2 < b_2 \le \ldots < a_k \le b_k = 1$,

$$(W_n(b_1)-W_n(a_1),\ W_n(b_2)-W_n(a_2),\ \ldots,\ W_n(b_k)-W_n(a_k))$$

will also converge to the corresponding finite dimensional limit

$$(W(b_1)-W(a_1),\ W(b_2)-W(a_2),\ \ldots\ ,\ W(b_k)-W(a_k))$$

of an uncorrelated k-dimensional normal distribution. $W_n(a)$ is tight, by Lemma 1. Therefore, $W_n(a)$ converges weakly to the corresponding limit distribution. This limit distribution is the Wiener process $W(a)$, using the result of Billingsley (1968).

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